

Exercise 1.1 :

We consider a random variable x on $] - \infty, +\infty[$ following a Gauss's law with μ and σ parameters.

- Give the expression of Gauss's law.
- Give the conditions on parameters to make this law be a probability distribution function (pdf).
- What is the Characteristic function (definition and expression)?
- Do the same for cumulative function.

Reminder :

$$\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z \exp(-t^2) dt \quad (1)$$

Correction :

Let x be a random variable defined on $] - \infty, +\infty[$ and following a Normal distribution (Gaussian).

Distribution function is :

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right) \quad (2)$$

Conditions and parameters to make this formula be a pdf :

- $f(x) \geq 0$
- $\forall x \int_{-\infty}^{+\infty} f(x) dx = 1$
- $\mu, \sigma \in \mathbb{R}$ et $\sigma > 0$

Characteristic function :

$$\Phi(t) = \int_{-\infty}^{+\infty} f(x) e^{itx} dx \quad (3)$$

$$\implies \Phi(t) = \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{+\infty} \exp\left(-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2 + itx\right) dx \quad (4)$$

We do the following substitution variable :

$$x = \sqrt{2}\sigma X + \mu \implies dx = \sqrt{2}\sigma dX \quad (5)$$

$$\Rightarrow \Phi(t) = \frac{1}{\sqrt{\pi}} e^{it\mu} \int_{-\infty}^{+\infty} e^{-X^2} e^{it\sqrt{2}\sigma X} dX \quad (6)$$

We use this equality (demonstration can be found on Web) :

$$\int_{-\infty}^{+\infty} e^{-t^2} e^{zt} dt = \sqrt{\pi} e^{z^2/4} \quad (7)$$

Taking $z = it\sqrt{2}\sigma$, one gets :

$$\Phi(t) = \exp(it\mu) \exp\left(-\frac{t^2\sigma^2}{2}\right) \quad (8)$$

Cumulative function :

$$F(x) = P(X \leq x) = \int_{-\infty}^x f(x') dx' = \int_{-\infty}^{\mu} f(x') dx' + \int_{\mu}^x f(x') dx' \quad (9)$$

$$F(x) = \int_{-\infty}^{\mu} \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{1}{2}\left(\frac{x' - \mu}{\sigma}\right)^2\right) dx' + \int_{\mu}^x \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{1}{2}\left(\frac{x' - \mu}{\sigma}\right)^2\right) dx' \quad (10)$$

Another substitution variable $x'' = x' - \mu$ gives :

$$\int_{-\infty}^0 \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{1}{2}\left(\frac{x''}{\sigma}\right)^2\right) dx'' = \frac{1}{2} \quad (11)$$

For the second integral, doing the substitution $v = \frac{x''}{\sigma}$, we have :

$$\frac{1}{\sqrt{\pi}} \int_0^{\frac{x-\mu}{\sqrt{2}\sigma}} e^{-v^2} dv = \frac{1}{2} \operatorname{erf}\left(\frac{x-\mu}{\sqrt{2}\sigma}\right) \quad (12)$$

So the final result :

$$F(x) = \frac{1}{2} + \frac{1}{2} \operatorname{erf}\left(\frac{x-\mu}{\sqrt{2}\sigma}\right) \quad (13)$$

Exercise 1.2 :

We do another substitution variable $y = \exp(x)$, where x is defined above and follows Gauss's law.

- How will you calculate the pdf of y ?
- Give the expression of pdf(y).

- Calculate the expectation $E(y)$; first write its definition.
- Calculate the expression of $V(y)$ (variance of y).

Correction :

We have to find the distribution function of y recorded $g(y)$. Then,

$$(x \in]-\infty, +\infty[) \Rightarrow (y = e^x \in]0, +\infty[) \quad (14)$$

Transfert theorem : taking into account that $x = \ln y$ and there is conservation of probabilities with variable substitution, we have :

$$f(x)dx = g(y)dy \quad (15)$$

that implies :

$$g(y) = f(x(y)) \left| \frac{dx}{dy} \right| \quad (16)$$

So we deduce for $g(y)$:

$$g(y) = \begin{cases} \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{1}{2}\left(\frac{\ln y - \mu}{\sigma}\right)^2\right) \frac{1}{y} & \text{if } y > 0 \\ 0 & \text{if } y \leq 0 \end{cases} \quad (17)$$

Expectation $E(y)$:

$$E(y) = \int_{-\infty}^{+\infty} y g(y) dy = \int_0^{+\infty} y g(y) dy \quad (18)$$

with $g(y) = 0$ if $y \leq 0$.

Given $x = \ln y$ and $dy = e^x dx$, we get :

$$E(y) = \int_{-\infty}^{+\infty} e^x \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{1}{2}\left(\frac{x - \mu}{\sigma}\right)^2\right) dx \quad (19)$$

One recognizes (6) equation for characteristic function :

$$E(y) = \Phi(it = 1) = e^{\mu + \frac{\sigma^2}{2}} \quad (20)$$

Variance $V(y)$:

$$V(y) = E(y^2) - E(y)^2 \quad (21)$$

$$E(y^2) = \int_{-\infty}^{+\infty} y^2 g(y) dy \quad (22)$$

Following the same previous reasoning :

$$E(y^2) = \int_{-\infty}^{+\infty} e^{2x} \frac{1}{\sqrt{2\pi\sigma}} \exp\left(-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right) dx \quad (23)$$

$$E(y^2) = \Phi(it = 2) \quad (24)$$

Finally :

$$V(y) = \exp(2\mu + 2\sigma^2) - \exp(2\mu + \sigma^2) \quad (25)$$

Exercise 1.3 :

We have a set of random variables $x_i, i = 1, \dots, n$ and we consider the random variable $z = \sum_{i=1}^n a_i x_i$ where a_i are constants.

- Write the expectation $E(z)$.
- Write the expression of $V(z)$ (in general case).

Correction :

Expectation $E(z)$:

$$E(z) = E\left(\sum_{i=1}^n a_i x_i\right) = \sum_{i=1}^n a_i E(x_i) \quad (26)$$

Variance $V(z)$:

$$V(z) = \sum_{i=1}^n a_i^2 V(x_i) + \sum_{i \neq j} a_i a_j \text{cov}(x_i, x_j) \quad (27)$$

with $\text{cov}(X, Y) = E[(X - E(X))(Y - E(Y))]$

Exercise 2.1 :

One starts from 2 random variables x and y with respectively σ_x^2 and σ_y^2 standard deviations and ρ_{xy} correlation factor.

We do the following substitution of variable :

$$\begin{aligned} - u &= x + y\sigma_x/\sigma_y \\ - v &= x - y\sigma_x/\sigma_y \end{aligned}$$

Calculate $V(u)$ and $V(v)$ variances and covariance $cov(u, v)$. What can you conclude ? What is the correlation factor ρ_{uv} ?

Correction :

We use the previous relation (27) so that :

$$V(u) = V(x) + \left(\frac{\sigma_x}{\sigma_y}\right)^2 V(y) + 2\frac{\sigma_x}{\sigma_y} cov(x, y) \quad (28)$$

$$\Rightarrow V(u) = 2\sigma_x^2 + 2\sigma_x^2\rho_{xy} = 2\sigma_x^2(1 + \rho_{xy}) \quad (29)$$

In the same way for $V(v)$:

$$\Rightarrow V(v) = 2\sigma_x^2 - 2\sigma_x^2\rho_{xy} = 2\sigma_x^2(1 - \rho_{xy}) \quad (30)$$

$$\text{Reminder : } \rho_{xy} = \frac{cov(x, y)}{\sigma_x\sigma_y}$$

We can get for variances calculation of u and v :

$$cov(u, v) = E((u - \bar{u})(v - \bar{v})) = E\left(\left(x + y\frac{\sigma_x}{\sigma_y} - \bar{x} - \bar{y}\frac{\sigma_x}{\sigma_y}\right)\left(x - y\frac{\sigma_x}{\sigma_y} - \bar{x} + \bar{y}\frac{\sigma_x}{\sigma_y}\right)\right) \quad (31)$$

$$= E\left(\left(x - \bar{x}\right)^2 - \frac{\sigma_x}{\sigma_y}(y - \bar{y})(x - \bar{x}) + (x - \bar{x})(y - \bar{y})\frac{\sigma_x}{\sigma_y} - \left(\frac{\sigma_x}{\sigma_y}\right)^2(y - \bar{y})^2\right) \quad (32)$$

$$= V(x) - \left(\frac{\sigma_x}{\sigma_y}\right)^2 V(y) = 0 \quad (33)$$

$$\Rightarrow \rho_{uv} = 0 \quad (34)$$

We see that u and v are not correlated : conclusion is that we can made 2 no-correlated variables from 2 correlated ones.

Exercise 2.2 :

Both x and y random variables follow a Gauss's law (pdf= $f(x, y)$) with $\mu_x = 0$ and $\mu_y = 0$.

- Write its expression.
- Calculate Expectations $E(x)$ and $E(y)$.
- Calculate the distribution function for u and v (pdf= $h(u, v)$).
- What do you conclude about these two random variables u and v ?

Correction :

General formula of 2D Gaussian distribution function :

$$f(x, y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2(1-\rho^2)}\left(\left(\frac{x-\mu_x}{\sigma_x}\right)^2 + \left(\frac{y-\mu_y}{\sigma_y}\right)^2 - \frac{2\rho(x-\mu_x)(y-\mu_y)}{\sigma_x\sigma_y}\right)\right) \quad (35)$$

In our case, function is written with $\mu_x = \mu_y = 0$:

$$f(x, y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2(1-\rho^2)}\left(\left(\frac{x}{\sigma_x}\right)^2 + \left(\frac{y}{\sigma_y}\right)^2 - \frac{2\rho xy}{\sigma_x\sigma_y}\right)\right) \quad (36)$$

Expectations :

$$E(x) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} x f(x, y) dx dy = \mu_x = 0 \quad (37)$$

$$E(y) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} y f(x, y) dx dy = \mu_y = 0 \quad (38)$$

Calculate now the $h(u, v)$ pdf : from the transfert theorem, there is probabilities conservation for (x, y) and (u, v) couples :

$$h(u, v) dudv = f(x, y) dx dy \quad (39)$$

then :

$$h(u, v) = f(\Phi(u, v)) | \det(J_\Phi(u, v)) | \quad (40)$$

where $\Phi : (u, v) \longrightarrow (x, y)$ et $J_\Phi(u, v)$ is the Jacobian of Φ transformation.

$$\det(J_\Phi(u, v)) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \quad \text{with } x = \frac{u+v}{2} \text{ et } y = \frac{u-v}{2} \frac{\sigma_y}{\sigma_x} \quad (41)$$

$$\det(J_\Phi(u, v)) = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} \frac{\sigma_y}{\sigma_x} & -\frac{1}{2} \frac{\sigma_y}{\sigma_x} \end{vmatrix} = -\frac{1}{2} \frac{\sigma_y}{\sigma_x} \quad (42)$$

Finally, one deduces $h(u, v)$ pdf :

$$h(u, v) = \frac{1}{2} \frac{\sigma_y}{\sigma_x} \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2(1-\rho^2)} \left(\left(\frac{u+v}{2}\right)^2 \frac{1}{\sigma_x^2} - \frac{1}{2} \left(\frac{(u-v)\sigma_y}{2\sigma_x}\right)^2 \frac{1}{\sigma_y^2} - 2\rho \frac{(u+v)(u-v)\sigma_y}{4\sigma_x\sigma_y} \frac{\sigma_y}{\sigma_x} \right)\right) \quad (43)$$

$$h(u, v) = \frac{1}{4} \frac{1}{\pi\sigma_x^2\sqrt{1-\rho^2}} \exp\left(-\frac{1}{8(1-\rho^2)} \left(\frac{u^2+2uv+v^2}{\sigma_x^2} \right) - \frac{1}{8(1-\rho^2)} \left(\frac{u^2-2uv+v^2}{\sigma_x^2} \right) + \frac{\rho}{4(1-\rho^2)} \left(\frac{u^2-v^2}{\sigma_x^2} \right)\right) \quad (44)$$

$$h(u, v) = \frac{1}{4} \frac{1}{\pi\sigma_x^2\sqrt{1-\rho^2}} \exp\left(-\frac{1}{4(1-\rho^2)} \frac{u^2(1-\rho)}{\sigma_x^2} - \frac{1}{4(1-\rho^2)} \frac{v^2(1+\rho)}{\sigma_x^2}\right) \quad (45)$$

Since we get from Exercise 2.1 :

$$\sigma_x^2(1 + \rho_{xy}) = \frac{\sigma_u^2}{2}\sigma_x^2(1 - \rho_{xy}) = \frac{\sigma_v^2}{2} \quad (46)$$

then,

$$h(u, v) = \frac{1}{\sqrt{2\pi}\sigma_u} \exp\left(-\frac{u^2}{2\sigma_u^2}\right) \frac{1}{\sqrt{2\pi}\sigma_v} \exp\left(-\frac{v^2}{2\sigma_v^2}\right) \quad (47)$$

$$h(u, v) = \text{Gauss}(u, 0, \sigma_u) \cdot \text{Gauss}(v, 0, \sigma_v) \quad (48)$$

We can see that u and v are independent because of pdf factorization $h(u, v) = \text{pdf}(u) \cdot \text{pdf}(v)$. If two variables X and Y are statistically independent, then they are not correlated. The reverse is not necessarily true. Indeed, no-correlation implies independence only in particular cases and Gaussian random variables is one of this special cases.

Exercise 3.1 :

One considers a continuous and positive random variable t , following an exponential law $\propto \exp(-t/\tau)$. Give the expression of its pdf $f(t)$ and calculate its expectation $E(t)$ and variance $V(t)$. We have N measures t_i of this variable and choose the maximum likelihood method to determine τ parameter. Describe the principle of the maximum likelihood method and apply it to calculate this parameter and its variance. Now, we have 2 independent samples of N_1 and N_2 measures so that total sample is consisted of $N = N_1 + N_2$ measures. Calculate the estimation of τ from these N measures while making appear the contribution of τ_1 and τ_2 .

Correction :

Let t be a random variable > 0 following an exponential law ; pdf is given by :

$$f(t) = \begin{cases} \frac{1}{\tau} \exp\left(-\frac{t}{\tau}\right) & \text{if } t \geq 0 \\ 0 & \text{if } t < 0 \end{cases} \quad (49)$$

Moreover :

$$\int_0^{+\infty} f(t)dt = 1 \text{ and } f(t) \geq 0 \forall t \quad (50)$$

To get expectation, we use integration rules :

$$E(t) = \int_0^{+\infty} \frac{t}{\tau} \exp\left(-\frac{t}{\tau}\right) dt = \left[\frac{t}{\tau} (-\tau) \exp\left(-\frac{t}{\tau}\right)\right]_0^{+\infty} - \int_0^{+\infty} \frac{1}{\tau} (-\tau) \exp\left(-\frac{t}{\tau}\right) dt \quad (51)$$

$$= 0 + \int_0^{+\infty} \exp\left(-\frac{t}{\tau}\right) dt = \tau \quad (52)$$

For variance :

$$E(t^2) = \int_0^{+\infty} \frac{t^2}{\tau} \exp\left(-\frac{t}{\tau}\right) dt = \left[-t^2 \exp\left(-\frac{t}{\tau}\right)\right]_0^{+\infty} - \int_0^{+\infty} \frac{1}{\tau} \frac{2t(-\tau)}{\tau} \exp\left(-\frac{t}{\tau}\right) dt \quad (53)$$

$$= 0 + 2 \int_0^{+\infty} t \exp\left(-\frac{t}{\tau}\right) dt = 2\tau^2 \quad (54)$$

$$\Rightarrow V(t) = \tau^2 \Rightarrow \sigma = \tau$$

We own N measures $t_i, i = 1, \dots, N$. Maximum likelihood allows to compute an estimation of τ parameter which is solution of the maximum or minimum for likelihood function \mathcal{L} defined as :

$$\mathcal{L} = \prod_{i=1}^N f(t_i) \quad (55)$$

where $f(t)$ is the pdf of t variable. τ parameter is found minimizing \mathcal{L} :

$$\frac{\partial(-\ln(\mathcal{L}))}{\partial\tau} = 0 \quad (56)$$

Calculate in our case the likelihood function \mathcal{L} :

$$\mathcal{L} = \prod_{i=1}^N \left(\frac{1}{\tau} e^{-\frac{t_i}{\tau}}\right) \quad (57)$$

We get :

$$-\ln \mathcal{L} = -\sum_{i=1}^N \ln\left(\frac{1}{\tau} e^{-\frac{t_i}{\tau}}\right) = N \ln(\tau) + \sum_{i=1}^N \frac{t_i}{\tau} \quad (58)$$

So one has to solve :

$$\frac{\partial -\ln \mathcal{L}}{\partial\tau} = 0 = \frac{N}{\tau} - \frac{1}{\tau^2} \sum_{i=1}^N t_i \quad (59)$$

$$\Rightarrow \tau = \frac{1}{N} \sum_{i=1}^N t_i \quad (60)$$

Variance of this estimation will be given by the second derivate :

$$V_\tau = -E\left(\frac{\partial^2(-\ln \mathcal{L})}{\partial \tau^2}\right)^{-1} \quad (61)$$

$$\begin{aligned} \Leftrightarrow V_\tau &= -E\left(\frac{\partial^2}{\partial \tau^2}\left(\sum_{i=1}^N \ln \tau + \sum_{i=1}^N \frac{t_i}{\tau}\right)\right)^{-1} \\ \Leftrightarrow V_\tau &= -E\left(\frac{\partial}{\partial \tau}\left(\frac{N}{\tau} - \frac{1}{\tau^2} \sum_{i=1}^N t_i\right)\right)^{-1} \\ &= -E\left(-\frac{N}{\tau^2} + 2\frac{\sum_{i=1}^N t_i}{\tau^3}\right)^{-1} \\ &= E\left(\frac{N}{\tau^2}\right)^{-1} \\ \Rightarrow \sigma_\tau &= \frac{\tau}{\sqrt{N}} \end{aligned} \quad (62)$$

Now we have 2 independant samples N_1 et N_2 . Each produces its estimation :

$$\tau_1 = \frac{1}{N_1} \sum_{i=1}^{N_1} t_i \quad \sigma_{\tau_1} = \frac{\tau_1}{\sqrt{N_1}} \quad \tau_2 = \frac{1}{N_2} \sum_{j=1}^{N_2} t_j \quad \sigma_{\tau_2} = \frac{\tau_2}{\sqrt{N_2}} \quad (63)$$

Taking into account the total sample N_1+N_2 :

$$\tau = \frac{1}{N_1 + N_2} \left(\sum_{i=1}^{N_1} t_i + \sum_{j=1}^{N_2} t_j \right) \quad (64)$$

$$\Rightarrow \tau = \frac{N_1 \tau_1 + N_2 \tau_2}{N_1 + N_2} \quad (65)$$

This result is the weighted average from these 2 samples.

Exercise 3.2 :

Two independant experiments have measured (τ_1, σ_1) and (τ_2, σ_2) with σ_i representing errors on measures.

(1) From these two measures, assuming errors are gaussian, we want to get the estimation of τ and its error (i.e with a combination of two measures).

- Which method do you use ?
- Calculate the estimation of τ and its error.

(2) From these two measures (τ_1, σ_1) and (τ_2, σ_2) :
 Define the equivalent number \tilde{N}_1 and \tilde{N}_2 of each measure ; give the relations defining them. We use the maximum likelihood method to find the estimation of τ from the definition of these 2 equivalent numbers. Calculate this estimation of τ in this case (Making appear τ_1, σ_1 et τ_2, σ_2 in the expression). Compare it to previous expression in (1).

Correction :

As the previous Exercise, we choose the maximum likelihood method with the pdf of 2 measures :

$$f(\tau, \sigma) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{1}{2} \frac{(\tau - \hat{\tau})^2}{\sigma^2}\right) \quad (66)$$

One has to maximize the likelihood function :

$$\mathcal{L} = \prod_{i=1}^2 \frac{1}{\sqrt{2\pi}\sigma_i} \exp\left(-\frac{1}{2} \frac{(\tau_i - \hat{\tau})^2}{\sigma_i^2}\right) \quad (67)$$

taking the following condition :

$$\frac{\partial(-\ln(\mathcal{L}))}{\partial\hat{\tau}} = 0 \quad (68)$$

We get :

$$\Rightarrow \hat{\tau} = \frac{\tau_1/\sigma_1^2 + \tau_2/\sigma_2^2}{1/\sigma_1^2 + 1/\sigma_2^2} \quad (69)$$

$\sigma_{\hat{\tau}}$ is deducted from second derivate :

$$\frac{1}{\sigma_{\hat{\tau}}} = \frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2} \quad (70)$$

For these both measures, equivalent number \tilde{N} is defined by :

$$\frac{\sigma_1}{\tau_1} = \frac{1}{\sqrt{\tilde{N}_1}} \quad \frac{\sigma_2}{\tau_2} = \frac{1}{\sqrt{\tilde{N}_2}} \quad (71)$$

This is the relative error of measure expressed by the statistical error due to the number of events.

If we apply the calculation of exercise 3.1 with the equivalent number \tilde{N} , we have :

$$\hat{\tau} = \frac{\tilde{N}_1 \tau_1 + \tilde{N}_2 \tau_2}{\tilde{N}_1 + \tilde{N}_2} \quad (72)$$

Finally :

$$\hat{\tau} = \frac{\tau_1 / (\sigma_1 / \tau_1)^2 + \tau_2 / (\sigma_2 / \tau_2)^2}{1 / (\sigma_1 / \tau_1)^2 + 1 / (\sigma_2 / \tau_2)^2} \quad (73)$$

In conclusion :

case (1) : weighted by the square of inverse error.

case (2) : weighted by the square of relative error.

Exercise 4.1 :

We are looking to compute the integral of a $f(x, y)$ function by Monte-Carlo method :

$$\int_{x=0}^{x=1} \int_{y=0}^{y=x} f(x, y) dx dy \quad (74)$$

For this, we own a random number and uniform generator between $[0, 1]$.

How will you proceed (Make a schema of the integration area on (x, y) plane) ?

Correction :

Reminder - Monte-Carlo method :

From the transfert theorem, we get the expression of expectation for a function g representing the random variable X as :

$$G = E(g(X)) = \int_a^b g(x) f_X(x) dx \quad (75)$$

where f_X is a pdf on $[a, b]$ interval. One usually takes a uniform distribution on $[a, b]$:

$$f_X(x) = \frac{1}{b - a} \quad (76)$$

Principle is to generate a sample (x_1, x_2, \dots, x_N) with law of X (so we calculate an estimator called "the estimator of Monte-Carlo" from this sample).

Law of large numbers assumes to build this estimator from the empirical average :

$$\tilde{g}_N = \frac{1}{N} \sum_{i=1}^N g(x_i), \quad (77)$$

which is, by the way, an unbiased estimator of expectation. This is the estimator of Monte-Carlo. We can clearly see that, by replacing sample by a set of values located on an integral support, and the function to integrate, we can build an approximation of its value, statistically made.

Thanks to the "uniform generator", Monte-Carlo method gives a numerical value of the integral noticed I . Area of integration is represented on the figure 1 below :

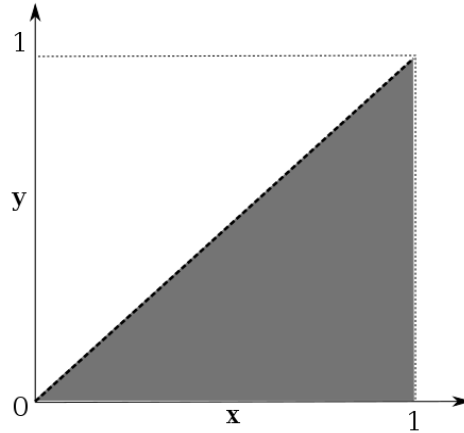


FIGURE 1 – Representation of integration area

So we can distinguish two cases :

case (1) - We do 2 random sampling, one for x and the other for y :

$$\text{Random sampling} = \begin{cases} x_i \in [0, 1] \\ y_i \in [0, 1] \end{cases} \quad (78)$$

If $x_i > y_i$, then we increment I in the following way : $I = I + f(x_i, y_i)$, else we redo a random sampling. We have done 2 random sampling and the average to lose one is $1/2$.

case (2) - We do 2 random sampling like previous :

$$\text{Random sampling} = \begin{cases} x_i \in [0, 1] \\ y_i \in [0, 1] \end{cases} \quad (79)$$

Then, as a function of results, we increment I in the following way :

$$\text{Incrementation} \begin{cases} \text{if } x_i > y_i & I = I + f(x_i, y_i) \\ \text{if } x_i < y_i & I = I + f(y_i, x_i) \end{cases} \quad (80)$$

The advantage here is that we use all the random sampling values unlike case (1). Finally, one has to multiply the I quantity by $(b - a)$ interval and divide by the size of random sampling N , so we get the numerical value of integral.

Exercise 4.2 :

A point source emits isotropically and covers an angle of θ_0 . A disk detector is positioned perpendicular to this source. So we have a cylindrical symmetry with 2 angles : φ between $[0, 2\pi]$ and θ as $\cos(\theta_0) < \cos(\theta) < 1$.

Calculate the pdf of $(\varphi, \cos \theta)$.

Having a generator of random values on $[0, 1]$, how will you random sample in acceptance disk a couple $(\varphi, \cos \theta)$?

With this detector, we want to do counting during equal interval time Δt . Express the distribution function for the number of recorded hits.

Correction :

Since point source emits isotropically, random variables φ and $\cos \theta$ follow a uniform law, respectively on $[0, 2\pi]$ et $[\cos \theta_0, 1]$.

We have for the φ pdf :

$$\text{pdf}(\varphi) = \frac{1}{2\pi} \quad (81)$$

For the $\cos \theta$ random variable, we can write :

$$\text{pdf}(\cos \theta) d \cos(\theta) = K d \cos(\theta) \quad \text{avec } K = \text{constante} \quad (82)$$

$$\Rightarrow \int_{\cos \theta_0}^1 \text{pdf}(\cos \theta) d \cos(\theta) = 1 = K \int_{\cos \theta_0}^1 d \cos(\theta) = K (1 - \cos \theta_0) \quad (83)$$

$$\Rightarrow K = \frac{1}{(1 - \cos \theta_0)} \quad (84)$$

Given φ and $\cos \theta$ are independant, we conclude :

$$\text{pdf}(\varphi, \cos \theta) = \text{pdf}(\varphi) \text{pdf}(\cos \theta) = \frac{1}{(1 - \cos \theta_0)} \frac{1}{2\pi} \quad (85)$$

With a random generator between $[0, 1]$, one makes the following correspondences :

$$\varphi = 2\pi u \quad \text{with } u \text{ uniform on } [0, 1] \quad (86)$$

For $\cos \theta$, we use the relation, taking random variable v as uniform on $[0, 1]$:

$$\int_{\cos \theta_0}^{\cos \theta} \frac{d \cos \theta}{1 - \cos \theta_0} = \int_0^v d v \quad (87)$$

\Rightarrow

$$\cos \theta - \cos \theta_0 = v(1 - \cos \theta_0) \Rightarrow \cos \theta = v + (1 - v)\cos \theta_0 \quad (88)$$

The number of hits recorded during interval Δt will follow a Poisson law.

Exercise 5.1 :

we have a set of measures y_i $i = 1, \dots, n$ depending from coordinates x_i and whose theoretical model is linear $y = ax + b$. Thanks to these data, we look for determining values of a and b .

Measures y_i have an error σ_i . Firstly, coordinates x_i are considered being without error.

- Express the χ^2 you have to use.
- Express the 2 equations from which you can deduce estimations for a and b .

Correction :

With n independant measures y_i $i = 1, \dots, n$ and n coordinates x_i in a linear model $y = ax + b$, with σ_i errors on y_i and no errors on x_i , one can write the χ^2 as :

$$\chi^2 = \sum_{i=1}^n \frac{(y_i - (ax_i + b))^2}{\sigma_i^2} \quad (89)$$

One has to minimize the χ^2 to compute a and b values : one gets 2 linear equations with 2 unknowns (a and b) :

$$\frac{\partial \chi^2}{\partial a} = \sum_{i=1}^n \frac{(y_i - ax_i - b)x_i}{\sigma_i^2} = 0 \quad (90)$$

and

$$\frac{\partial \chi^2}{\partial b} = \sum_{i=1}^n \frac{(y_i - ax_i - b)}{\sigma_i^2} = 0 \quad (91)$$

Exercise 5.2 :

Now, x_i coordinates have δ_i errors.

Express the χ^2 which has to be used in this case.

Write the 2 equations from which you calculate a and b estimation. What's the difference with previous case ?

Correction :

χ^2 formula must be modified because we take into account of δ_i errors on coordinates x_i . Indeed, variance of $(y_i - ax_i - b)$ is not only equal to $V(y_i) = \sigma_i^2$:

$$V(y_i - ax_i - b) = V(y_i) + a^2 V(x_i) = \sigma_i^2 + a^2 \delta_i^2 \quad (92)$$

The denominator of χ^2 depends on a parameter :

$$\chi^2 = \sum_{i=1}^n \frac{(y_i - (ax_i + b))^2}{\sigma_i^2 + a^2 \delta_i^2} \quad (93)$$

Minimization of χ^2 is got by the 2 following equations :

$$\frac{\partial \chi^2}{\partial a} = 0 \quad \text{et} \quad \frac{\partial \chi^2}{\partial b} = 0 \quad (94)$$

But we notice that all these equations are not linear since the second one which minimizes the χ^2 with b depends on powered a : we have not analytical solution in this case.

Exercise 6.1 :

χ^2 method can give estimations on parameters, $a \pm \sigma_a$ and $b \pm \sigma_b$, so a minimum value χ_{min}^2 .

We want to draw in a, b plane the contour related to a given confidence level.

Express the distribution function that you use.

In this case, by fixing the confidence level, express the variation of χ^2 compared to χ_{min}^2 , so we could write : $\chi^2(CL) = \chi_{min}^2 + \Delta\chi^2$

What value do you get with $CL = 0.68$?

Correction :

$\chi^2(CL)$ term only contains second derivatives of χ^2 at lowest level :

$$\frac{\partial^2 \chi^2}{\partial a^2}, \quad \frac{\partial^2 \chi^2}{\partial b^2} \quad \text{et} \quad \frac{\partial^2 \chi^2}{\partial a \partial b} \quad (95)$$

Concerning $\Delta\chi^2$, distribution function is a χ^2 law with 2 freedom degrees ; pdf is written as :

$$f(\Delta\chi^2) = \frac{1}{2} e^{-\frac{\Delta\chi^2}{2}} \quad (96)$$

So for a fixed confidence level CL , we have :

$$1 - CL = \int_{\Delta\chi_{CL}^2}^{+\infty} \frac{1}{2} e^{-\frac{\Delta\chi^2}{2}} d\chi^2 = e^{-\frac{\Delta\chi_{CL}^2}{2}} \quad (97)$$

also : $\Delta\chi_{CL}^2 = -2 \ln(1 - CL)$.

For $CL = 0.68$, we have : $\Delta\chi_{CL}^2 = 2.28$